# Learning Robot Manipulation Tasks with Semi-Tied Gaussian Mixture Models

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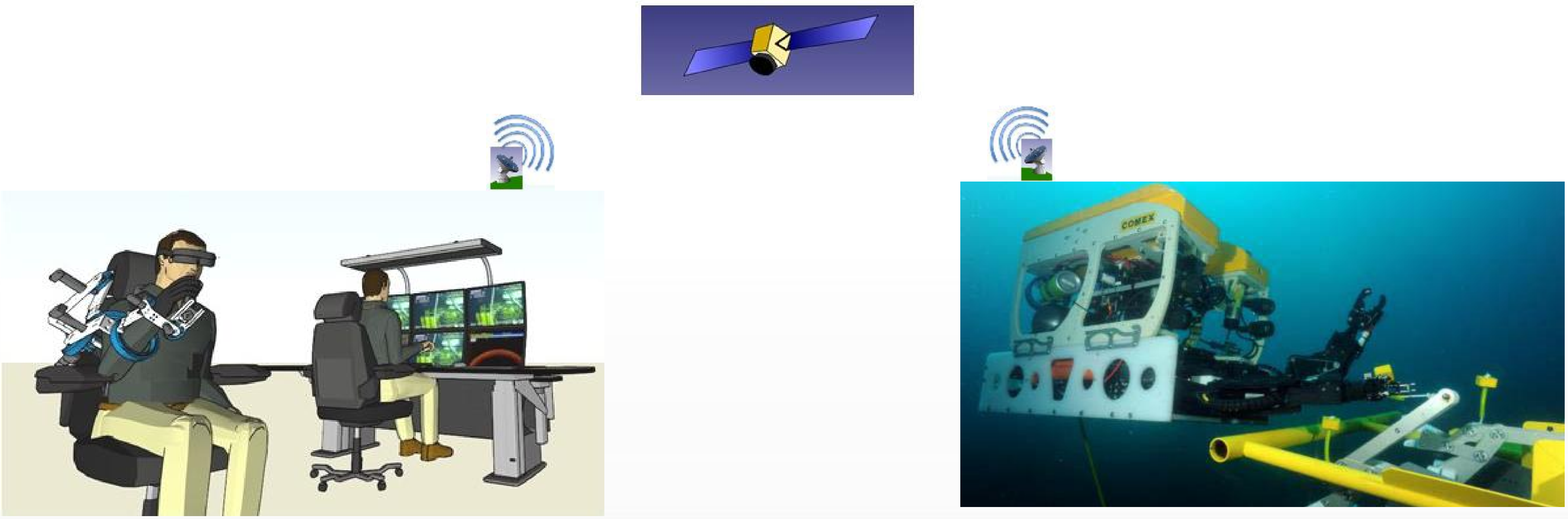
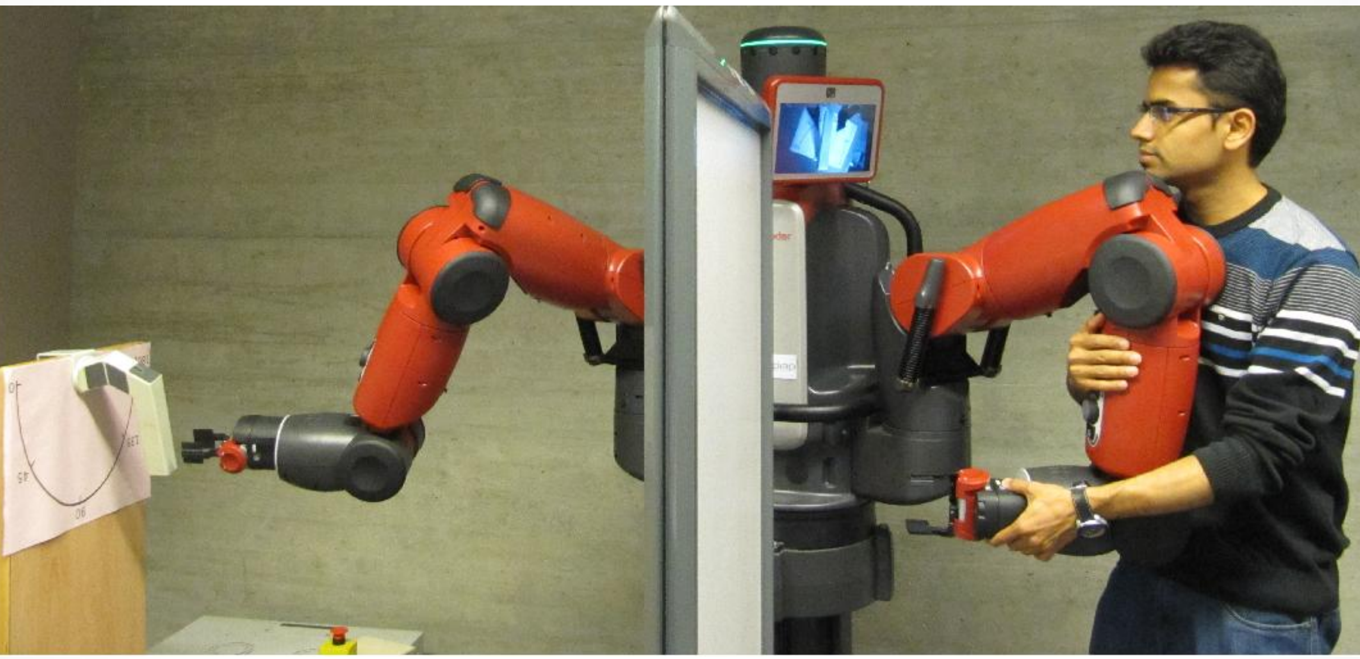
Swiss Machine Learning Day 2015

# Application – Skill Acquisition in Teleoperated Robots



# Semi-Autonomous Manipulation



Subspace Clustering

Task Adaptability

Autonomous Control

**Recognition of intentions on teleoperator side**

**Reproduction of movement on robot side**

# Outline

- **Semi-tied Gaussian mixture models**
- Task-parameterized semi-tied GMMs
- Hidden semi-Markov model encoding
- Linear quadratic tracking control
- Valve opening with Baxter robot

# Subspace Clustering

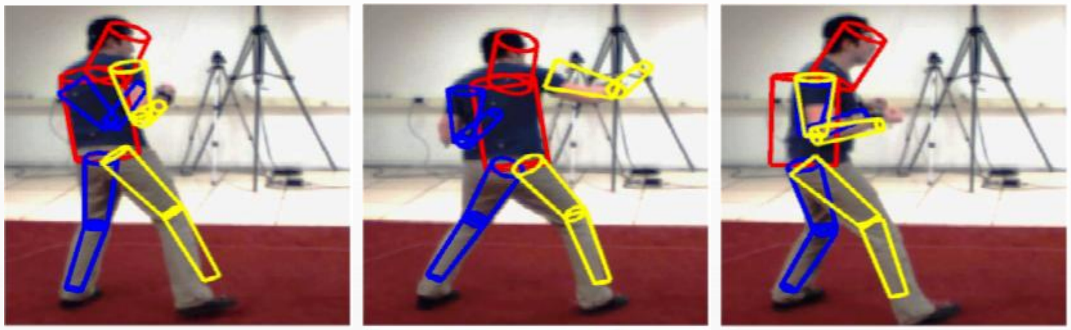
$$\{\boldsymbol{\xi}_t \in \mathbb{R}^D\}_{t=1}^T$$

$$\mathcal{P}(\boldsymbol{\xi}_t|\theta) = \sum_{i=1}^{K} \pi_i \, \mathcal{N}(\boldsymbol{\xi}_t|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i) \qquad \theta = \{\pi_i, \boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i\}_{i=1}^K$$

- Model over-fitting with $D \gg T$
- Need for parsimonious model with fewer parameters and better generalization
- Statistical subspace clustering imposes special structure on the covariance matrix to model the latent space of dimension $d$ with $d \ll D$
  - Isotropic, diagonal, block-diagonal, multiple diagonal, full

# Subspace Clustering

Motion segmentation and tracking



3D human motion tracking

[Elhamifar and Vidal, 2013] [Li *et al.*, 2009]

# Subspace Clustering

$$\mathcal{P}(\boldsymbol{\xi}_t|\theta) = \sum_{i=1}^{K} \pi_i \, \mathcal{N}(\boldsymbol{\xi}_t|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$$

- Mixture of factor analyzers

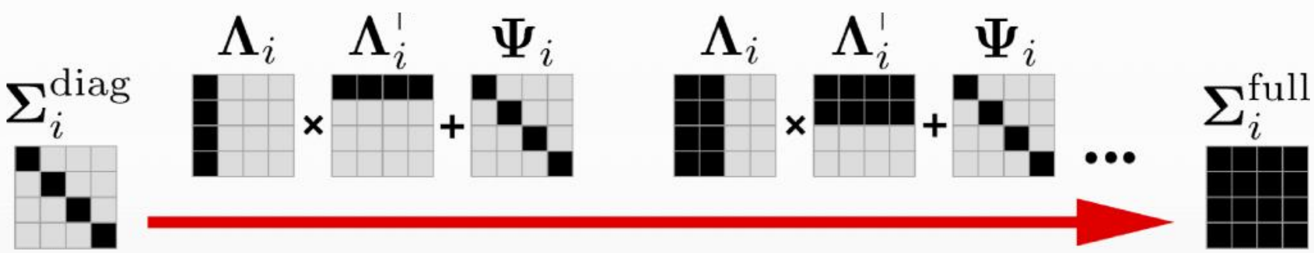
$$\boldsymbol{\Sigma}_i = \boldsymbol{\Lambda}_i \boldsymbol{\Lambda}_i^{\top} + \boldsymbol{\Psi}_i$$

- Probabilistic principal component analysis

$$\boldsymbol{\Sigma}_i = \boldsymbol{\Lambda}_i \boldsymbol{\Lambda}_i^{\top} + \sigma^2 \boldsymbol{I}_D$$

$\boldsymbol{\Lambda}_i \in \mathbb{R}^{D \times d} \Rightarrow$ factor loadings matrix

$\boldsymbol{\Psi}_i \in \mathbb{R}^{D \times D} \Rightarrow$ diagonal noise matrix



- Human movements are ***spatially*** and ***temporally correlated*** along ***important synergistic directions***
- Need for ***sharing the parameters*** across the mixture components

# Semi-Tied Gaussian Mixture Models

$$\mathcal{P}(\boldsymbol{\xi}_t|\theta) = \sum_{i=1}^{K} \pi_i \, \mathcal{N}(\boldsymbol{\xi}_t|\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$$

$$\boldsymbol{\Sigma}_i = \mathbf{H}\boldsymbol{\Sigma}_i^{(\mathrm{diag})}\mathbf{H}^\top$$

$\mathbf{H} \in \mathbb{R}^{D\times D} \Rightarrow$ common latent basis vectors

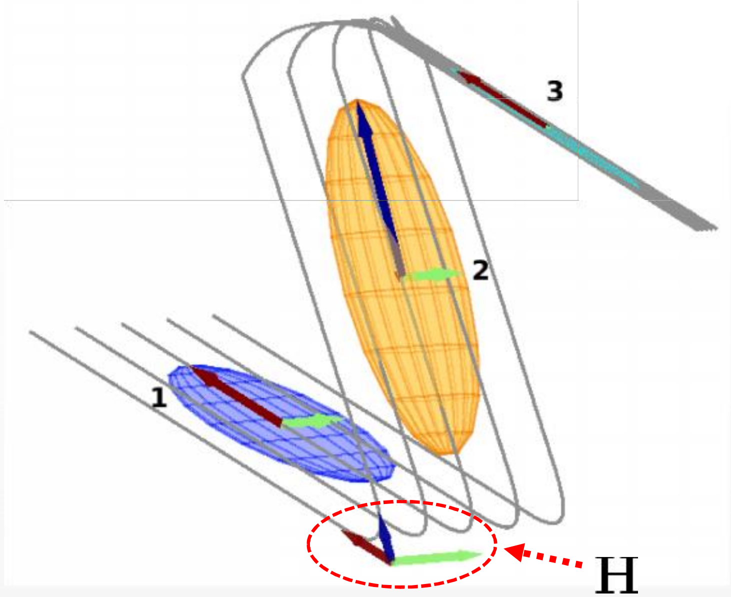
$\boldsymbol{\Sigma}_i^{(\mathrm{diag})} \in \mathbb{R}^{D\times D} \Rightarrow$ component-specific diagonal matrix

$\mathbf{S}_i \in \mathbb{R}^{D\times D} \Rightarrow$ empirical covariance matrix

- $\mathbf{H}$ applies global linear transformation to de-correlate the data, and $\boldsymbol{\Sigma}_i^{(\mathrm{diag})}$ selects the appropriate subspace

- Mixture components are aligned along the basis vectors for noisy and/or insufficient training data

$$\boldsymbol{\Sigma}_i := \alpha\mathbf{H}\boldsymbol{\Sigma}_i^{(\mathrm{diag})}\mathbf{H}^\top + (1-\alpha)\mathbf{S}_i \qquad \alpha \in (0,1)$$




[Gales, 1999]

# Semi-Tied Gaussian Mixture Models

$$\theta = \{\pi_i, \boldsymbol{\mu}_i, \mathbf{B}, \boldsymbol{\Sigma}_i^{(\mathrm{diag})}\}_{i=1}^{K}$$

$$\mathbf{B} = \mathbf{H}^{-1}$$

- E-Step:

$$h_{t,i}^{\hat{\theta}} := \frac{\pi_i \, \mathcal{N}(\boldsymbol{\xi}_t | \boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)}{\sum_{k=1}^{K} \pi_k \, \mathcal{N}(\boldsymbol{\xi}_t | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}$$

- M-Step:

$$\pi_i := \frac{\sum_{t=1}^{T} h_{t,i}}{T}$$

$$\boldsymbol{\mu}_i := \frac{\sum_{t=1}^{T} h_{t,i} \boldsymbol{\xi}_t}{\sum_{t=1}^{T} h_{t,i}}$$

$$\mathbf{S}_i := \frac{\sum_{t=1}^{T} h_{t,i} \, (\boldsymbol{\xi}_t - \boldsymbol{\mu}_i)(\boldsymbol{\xi}_t - \boldsymbol{\mu}_i)^{\top}}{\sum_{t=1}^{T} h_{t,i}}$$

Variational optimisation of $\boldsymbol{\Sigma}_i^{(\mathrm{diag})}$ and $\mathbf{B}$

$$\boldsymbol{\Sigma}_i^{(\mathrm{diag})} := \mathrm{diag}\left(\mathbf{B}\mathbf{S}_i\mathbf{B}^{\top}\right)$$

Cofactor matrix

$$\mathbf{C} := \mathbf{B}^{\top^{-1}} |\mathbf{B}|$$

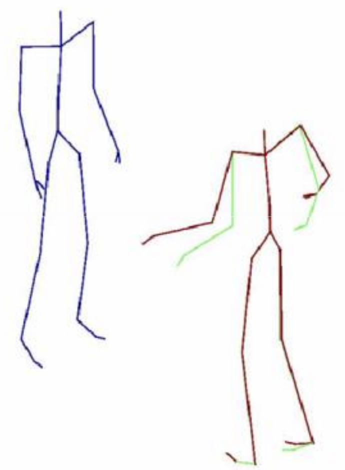
$$\mathbf{G}_d := \sum_{i=1}^{K} \frac{1}{\Sigma_{i,d}^{(\mathrm{diag})}} \mathbf{S}_i \sum_{t=1}^{T} h_{t,i}^{\hat{\theta}}$$

$$\mathbf{b}_d := \mathbf{c}_d \mathbf{G}_d^{-1} \sqrt{\frac{\sum_{t=1}^{T} \sum_{i=1}^{K} h_{t,i}^{\hat{\theta}}}{\mathbf{c}_d \mathbf{G}_d^{-1} \mathbf{c}_d^{\top}}}$$

$$\boldsymbol{\Sigma}_i := \alpha \mathbf{H} \boldsymbol{\Sigma}_i^{(\mathrm{diag})} \mathbf{H}^{\top} + (1-\alpha)\mathbf{S}_i$$

# Chicken Dance Encoding

$$D = 94, K = 75$$



- Regenerated movement sequence is shown in green

# Chicken Dance Encoding

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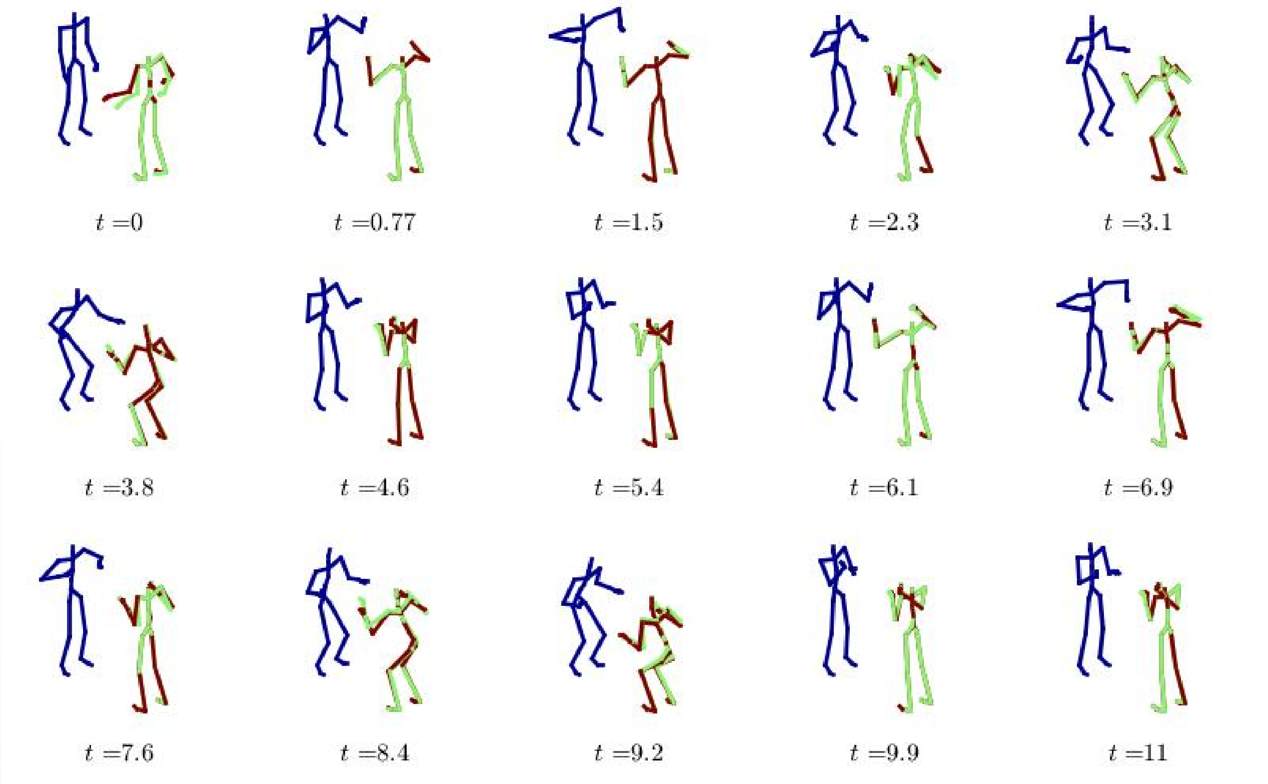


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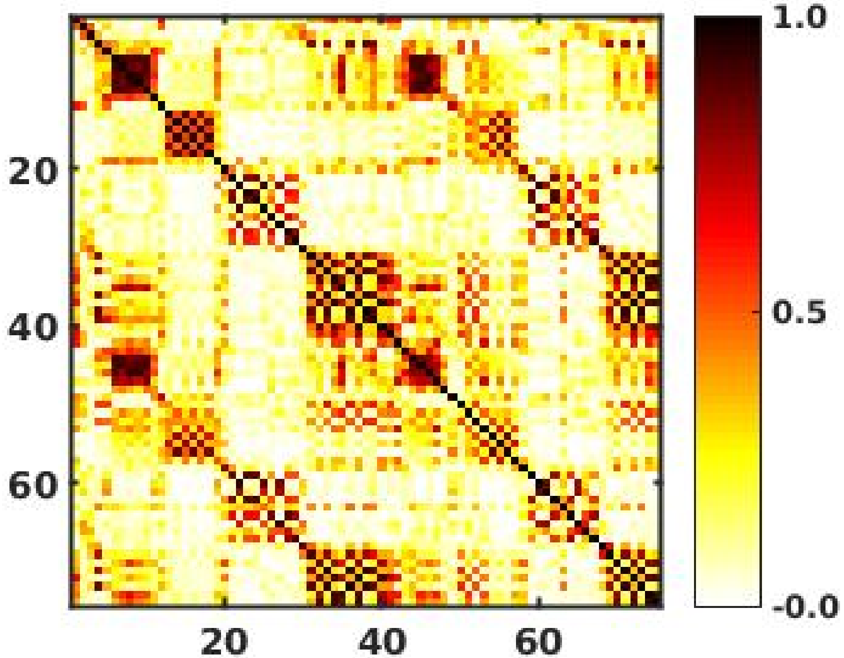
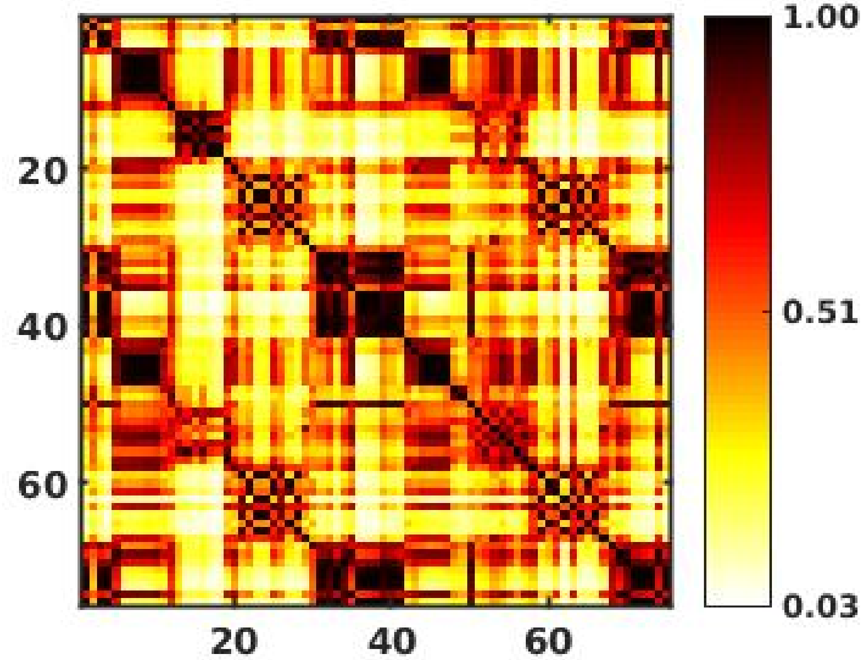


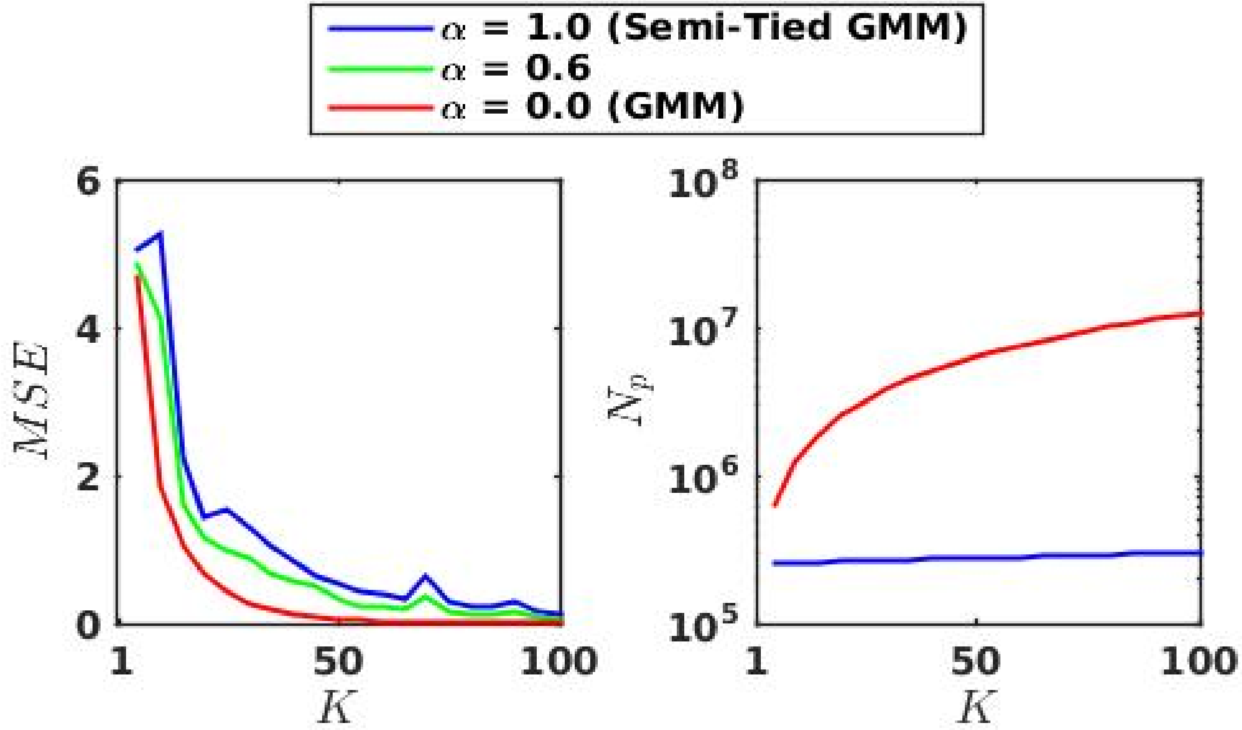

GMM




Semi-Tied GMM

- Semi-Tied GMM components are more correlated than standard GMM components

# Chicken Dance Encoding




- Semi-Tied model requires more components but the number of parameters remain less

# Outline

- Semi-tied Gaussian mixture models
- **Task-parameterized semi-tied GMMs**
- Hidden semi-Markov model encoding
- Linear quadratic tracking control
- Valve opening with Baxter robot

# Task-Parameterized Semi-Tied GMM

- Adopt the model parameters to new environmental situations using frames of reference
- Observe the data from $P$ coordinate systems $\{\boldsymbol{A}_j, \boldsymbol{b}_j\}_{j=1}^{P}$ : $\{\boldsymbol{\xi}_t^{(j)}\}_{j=1}^{P}$

$$\boldsymbol{\xi}_t^{(j)} = \boldsymbol{A}_j^{-1}(\boldsymbol{\xi}_t - \boldsymbol{b}_j)$$

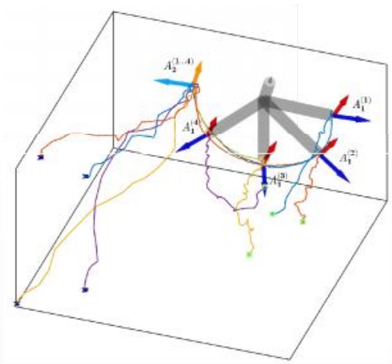
$$h_{t,i}^{\hat{\theta}_p} := \frac{\pi_i \, \mathcal{N}(\boldsymbol{\mu}_i^{(p)}, \boldsymbol{\Sigma}_i^{(p)})}{\sum_{k=1}^{K} \pi_k \, \mathcal{N}(\boldsymbol{\mu}_k^{(p)}, \boldsymbol{\Sigma}_k^{(p)})}$$

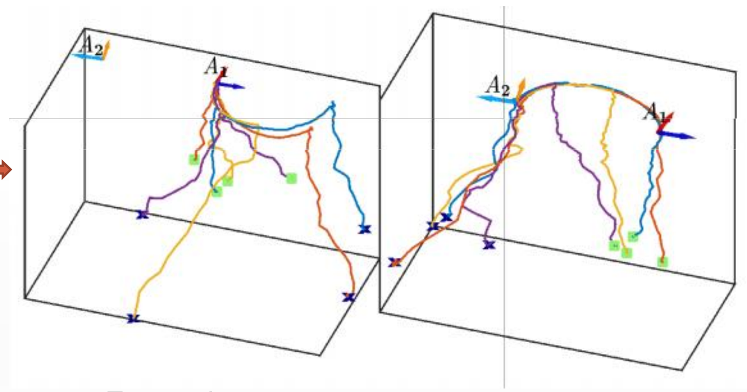
$$\mathcal{N}(\boldsymbol{\mu}_i^{(p)}, \boldsymbol{\Sigma}_i^{(p)}) = \prod_{j=1}^{P} \mathcal{N}\left(\boldsymbol{\xi}_t^{(j)} \,|\, \boldsymbol{\mu}_i^{(j)}, \boldsymbol{\Sigma}_i^{(j)}\right)$$

# Task-Parameterized Semi-Tied GMM

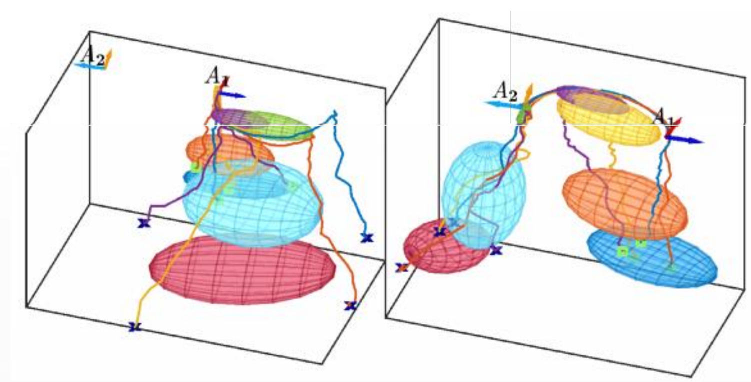
$$\boldsymbol{\xi}_t^{(j)} = \boldsymbol{A}_j^{-1}(\boldsymbol{\xi}_t - \boldsymbol{b}_j)$$

$$\theta_p = \{\pi_i, \{\boldsymbol{\mu}_i^{(j)}, \boldsymbol{\Sigma}_i^{(j)}\}_{j=1}^P\}_{i=1}^K$$





Frame 1 Frame 2

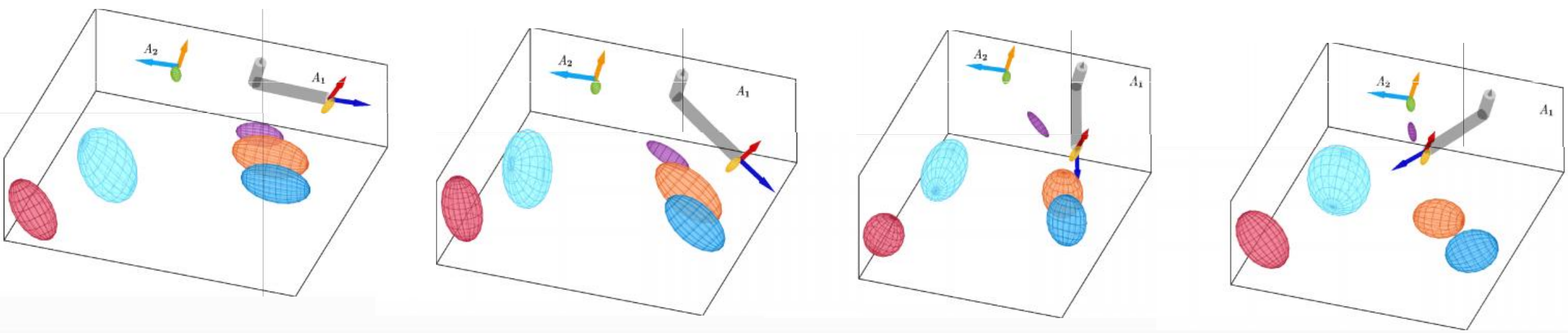


Frame 1 Frame 2

$$\boldsymbol{\Sigma}_i^{(j)} = \alpha \mathbf{H}^{(j)} \boldsymbol{\Sigma}_i^{(j)(\mathrm{diag})} \mathbf{H}^{(j)\top} + (1-\alpha)\mathbf{S}_i^{(j)}$$

# Task-Parameterized Semi-Tied GMM $\mathcal{N}(\tilde{\boldsymbol{\mu}}_i, \tilde{\boldsymbol{\Sigma}}_i) \propto \prod_{j=1}^{P} \mathcal{N}\left(\tilde{\boldsymbol{A}}_j \boldsymbol{\mu}_i^{(j)} + \tilde{\boldsymbol{b}}_j, \tilde{\boldsymbol{A}}_j \boldsymbol{\Sigma}_i^{(j)} \tilde{\boldsymbol{A}}_j^{\top}\right)$

- Given the new environmental situation $\{\tilde{\boldsymbol{A}}_j, \tilde{\boldsymbol{b}}_j\}_{j=1}^{P}$ , the model parameters are adapted by taking *product of linearly transformed Gaussians*




$$\tilde{\boldsymbol{\mu}}_i = \tilde{\boldsymbol{\Sigma}}_i \sum_{j=1}^{P} \left(\tilde{\boldsymbol{A}}_j \boldsymbol{\Sigma}_i^{(j)} \tilde{\boldsymbol{A}}_j^{\top}\right)^{-1} \left(\tilde{\boldsymbol{A}}_j \boldsymbol{\mu}_i^{(j)} + \tilde{\boldsymbol{b}}_j\right) \qquad \tilde{\boldsymbol{\Sigma}}_i = \left(\sum_{j=1}^{P} \left(\tilde{\boldsymbol{A}}_j \boldsymbol{\Sigma}_i^{(j)} \tilde{\boldsymbol{A}}_j^{\top}\right)^{-1}\right)^{-1}$$

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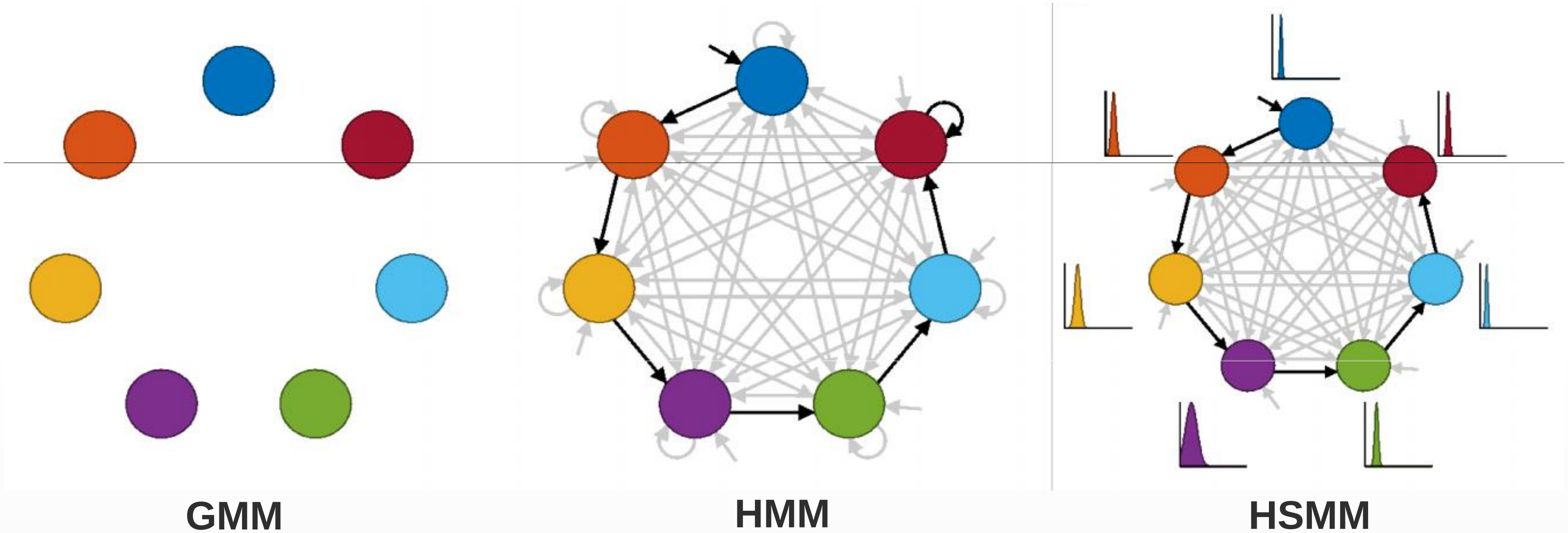
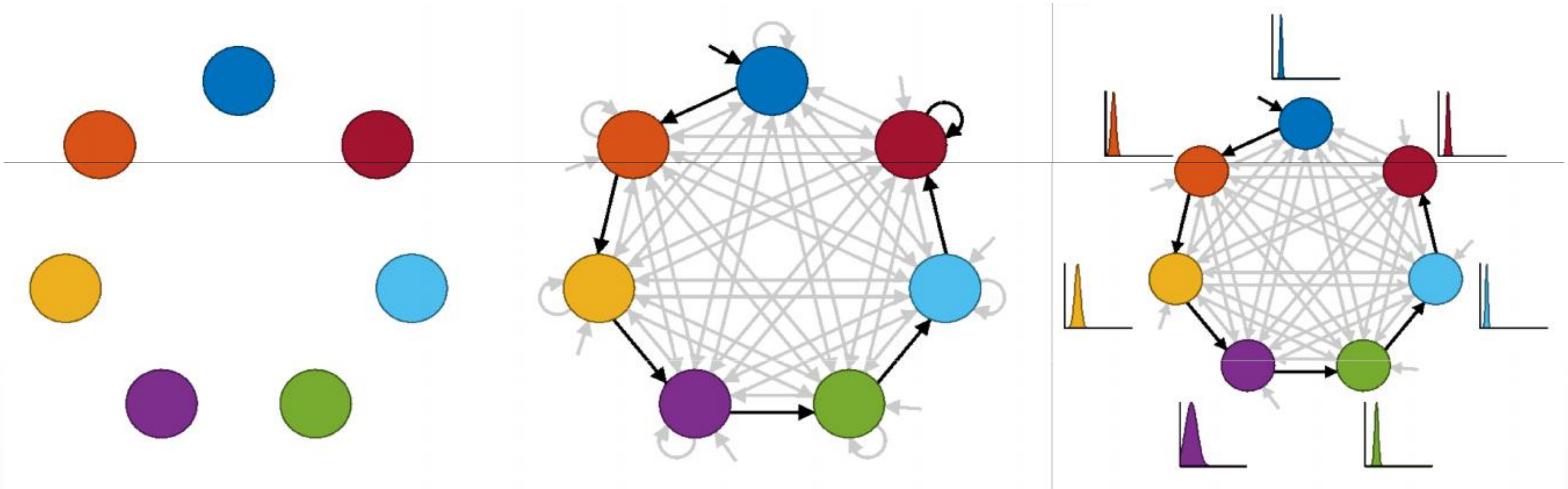
# Hidden Semi-Markov Model Encoding




- Recognize the current state of the task and re-plan the movement sequence
- Encapsulate the spatio-temporal information in the model

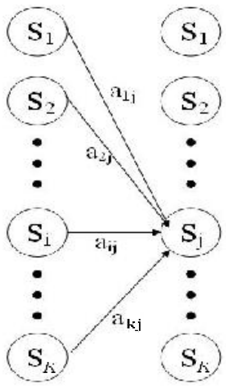
# Hidden Semi-Markov Model Encoding



- Each state output is a single Gaussian representing product of Gaussians
- Self-transition probability is explicitly modeled for state duration by a Gaussian

$$\theta_h = \left\{ \Pi_i, \{a_{i,m}\}_{m=1}^{K}, \{\boldsymbol{\mu}_i^{(j)}, \boldsymbol{\Sigma}_i^{(j)}\}_{j=1}^{P}, \mu_i^D, \Sigma_i^D \right\}_{i=1}^{K}$$

# Hidden Semi-Markov Model Encoding




- Generation of state sequence with datapoint $\xi_t$ to be in state $i$ at time $t$ is computed with forward variable

$$\alpha_{t,i}^{\text{HSMM}} = \sum_{j=1}^{K} \sum_{d=1}^{\min(d^{\max},t-1)} \alpha_{t-d,j}^{\text{HSMM}} \, a_{j,i} \mathcal{N}\left(d | \mu_i^D, \Sigma_i^D\right) \qquad \alpha_{1,i}^{\text{HSMM}} = \frac{\pi_i \mathcal{N}(\boldsymbol{\xi}_1 | \tilde{\boldsymbol{\mu}}_i, \tilde{\boldsymbol{\Sigma}}_i)}{\sum_{k=1}^{K} \pi_k \mathcal{N}(\boldsymbol{\xi}_1 | \tilde{\boldsymbol{\mu}}_k, \tilde{\boldsymbol{\Sigma}}_k)}$$

- Desired step-wise reference trajectory $\mathcal{N}(\hat{\boldsymbol{\mu}}_t, \hat{\boldsymbol{\Sigma}}_t)$ follows from the forward variable

$$q_t = \arg\max_i \; \alpha_{t,i}^{\text{HSMM}}, \quad \hat{\boldsymbol{\mu}}_t = \tilde{\boldsymbol{\mu}}_{q_t}, \quad \hat{\boldsymbol{\Sigma}}_t = \tilde{\boldsymbol{\Sigma}}_{q_t}$$

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# Linear Quadratic Tracking Control

- Desired step-wise reference trajectory $\mathcal{N}(\hat{\boldsymbol{\mu}}_t, \hat{\Sigma}_t)$ is smoothly tracked by minimizing the cost function starting from initial state $\boldsymbol{\xi}_1$

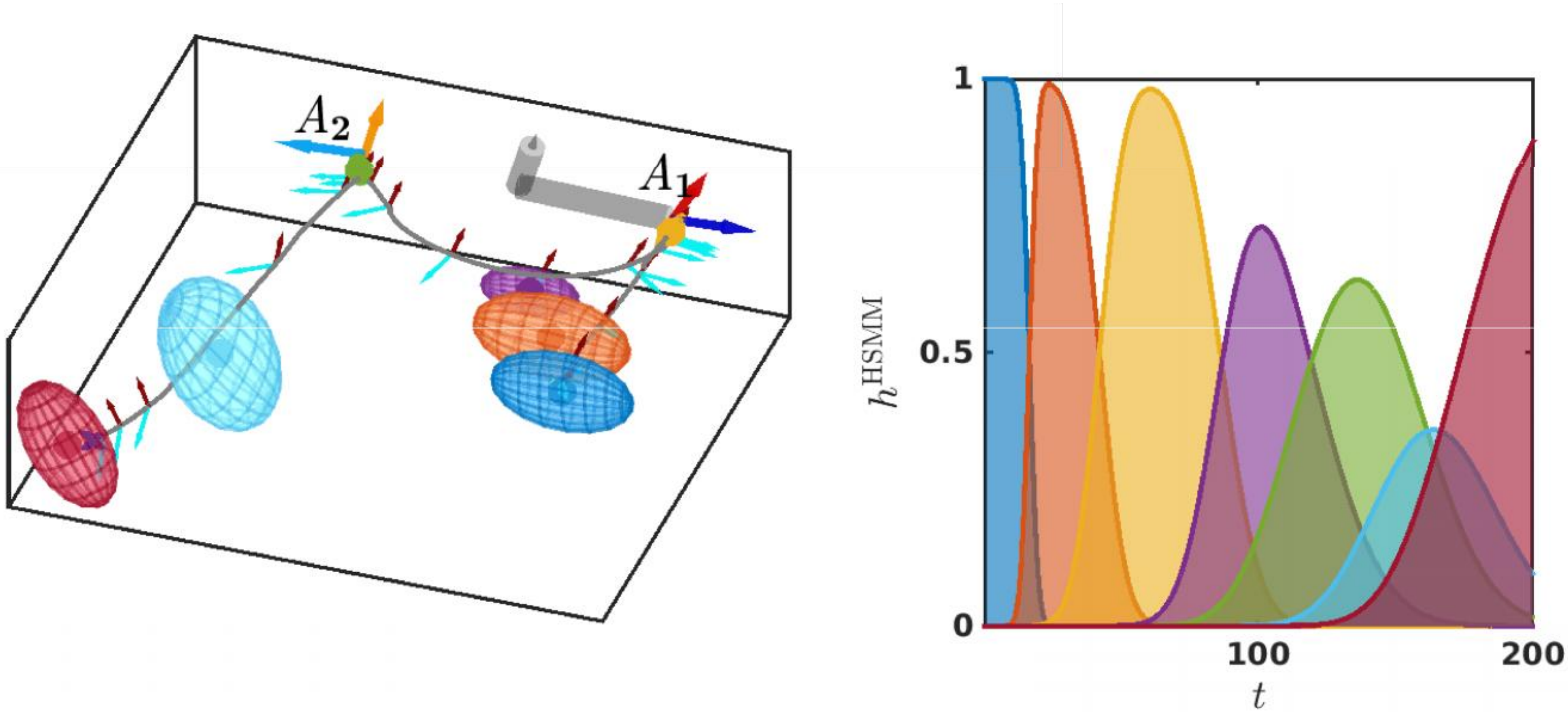
$$c_t(\boldsymbol{\xi}_t, \boldsymbol{u}_t) = \sum_{t=1}^{T} (\boldsymbol{\xi}_t - \hat{\boldsymbol{\mu}}_t)^{\top} \boldsymbol{Q}_t (\boldsymbol{\xi}_t - \hat{\boldsymbol{\mu}}_t) + \boldsymbol{u}_t^{\top} \boldsymbol{R}_t \boldsymbol{u}_t \qquad \boldsymbol{Q}_t = \hat{\boldsymbol{\Sigma}}_t^{-1} \succeq 0, \boldsymbol{R}_t \succ 0$$

$$\text{s.t.} \quad \dot{\boldsymbol{\xi}}_t = \boldsymbol{A}_d \boldsymbol{\xi}_t + \boldsymbol{B}_d \boldsymbol{u}_t$$

- Optimal control input is obtained by solving a set of differential equations

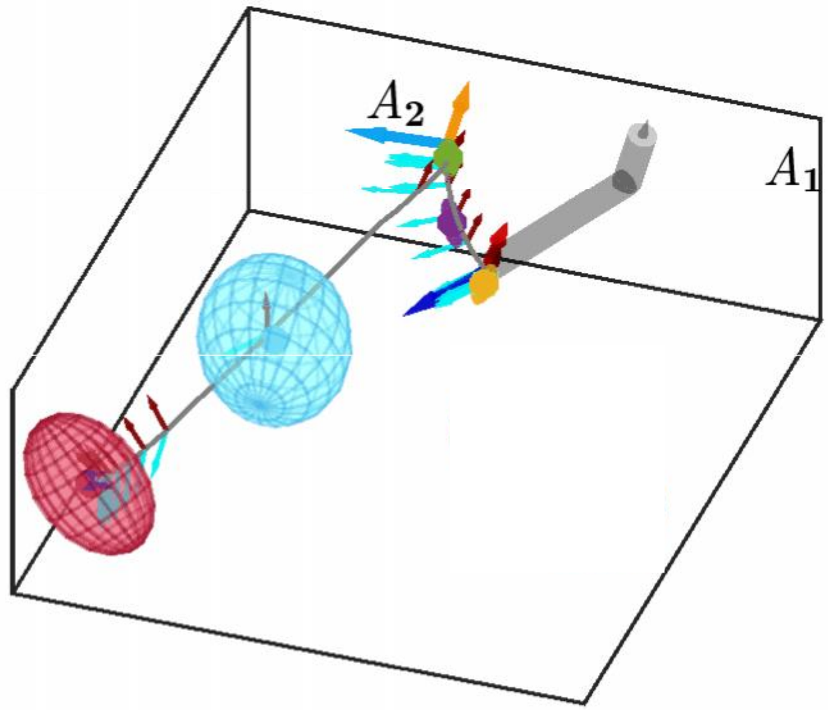
$$\boldsymbol{u}_t^* = \boldsymbol{K}_t^{\mathcal{P}} (\hat{\boldsymbol{\mu}}_t^x - \boldsymbol{x}_t) + \boldsymbol{K}_t^{\mathcal{V}} (\hat{\boldsymbol{\mu}}_t^{\dot{x}} - \dot{\boldsymbol{x}}_t) - \boldsymbol{R}_t^{-1} \boldsymbol{B}_d^{\top} \boldsymbol{d}_t \qquad \begin{aligned} \boldsymbol{\xi}_t &= [\boldsymbol{x}_t^{\top}\ \dot{\boldsymbol{x}}_t^{\top}]^{\top} \\ \hat{\boldsymbol{\mu}}_t &= [\hat{\boldsymbol{\mu}}_t^{x\top}\ \hat{\boldsymbol{\mu}}_t^{\dot{x}\top}]^{\top} \end{aligned}$$

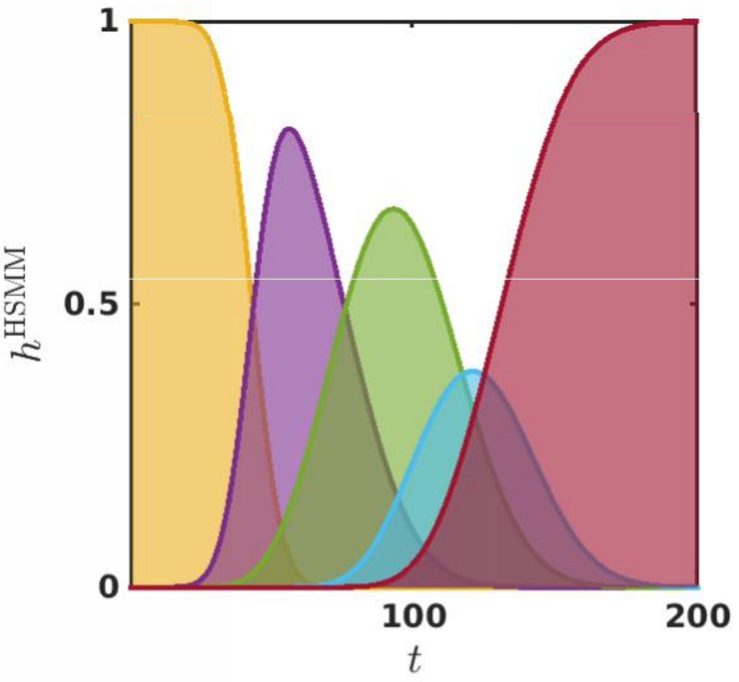
# Trajectory Reproduction




- Task variability is used for adjusting the compliance in following the trajectory

# Trajectory Reproduction




- Task variability is used for adjusting the compliance in following the trajectory

# Outline

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- **Valve opening with Baxter robot**

# Valve Opening Experiment with Baxter

$D = 14, K = 7$

- Two frames: $\{\boldsymbol{A}_1, \boldsymbol{b}_1\}$ for initial configuration of the valve, $\{\boldsymbol{A}_2, \boldsymbol{b}_2\}$ for desired end configuration of the valve
- Eight demonstrations $n = 1 \dots 8$ downsampled to 200 datapoints, 50-50 training testing ratio, and $D = 14$

$$\boldsymbol{A}_j^{(n)} = \begin{bmatrix} \boldsymbol{R}_j^{(n)} & & & \boldsymbol{0} \\ & \mathcal{E}_j^{(n)} & & \\ \boldsymbol{0} & & \boldsymbol{R}_j^{(n)} & \\ & & & \mathcal{E}_j^{(n)} \end{bmatrix}, \boldsymbol{b}_j^{(n)} = \begin{bmatrix} \boldsymbol{p}_j^{(n)} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

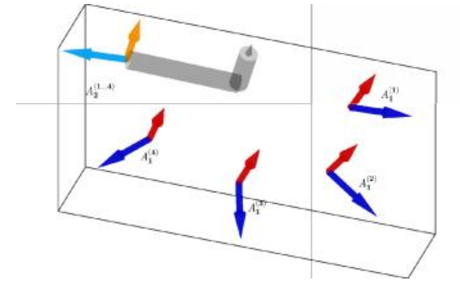
$\boldsymbol{x}_t^p \in \mathbb{R}^3 \Rightarrow$ Cartesian position

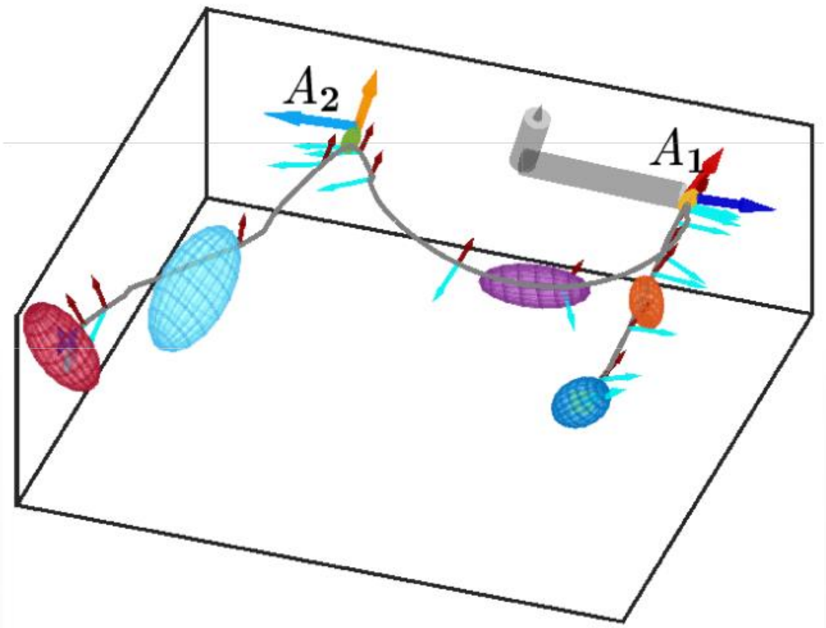
$\boldsymbol{\varepsilon}_t^o \in \mathbb{R}^4 \Rightarrow$ Quaternion orientation

$\dot{\boldsymbol{x}}_t^p \in \mathbb{R}^3 \Rightarrow$ Linear velocity

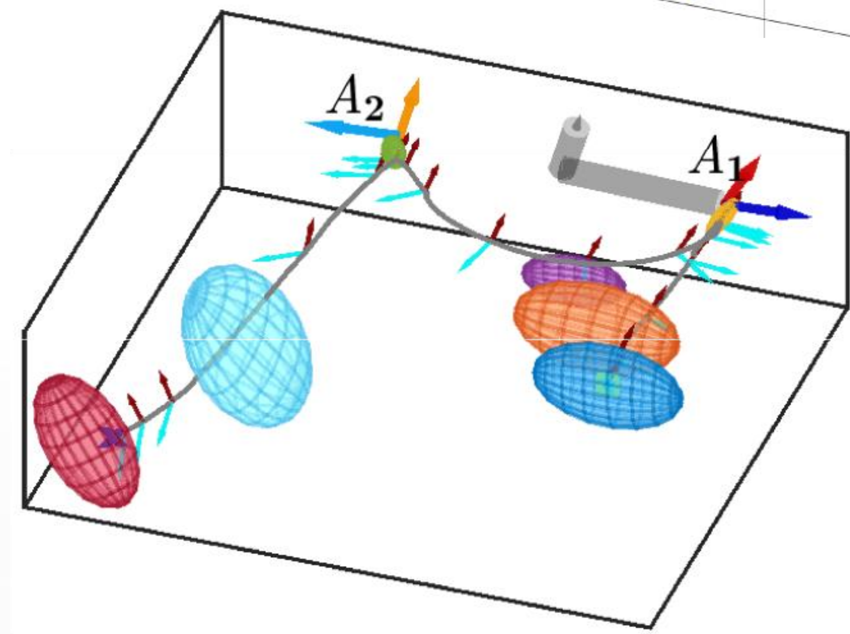
$\dot{\boldsymbol{\varepsilon}}_t^o \in \mathbb{R}^4 \Rightarrow$ Quaternion derivative

# Valve Opening Experiment with Baxter





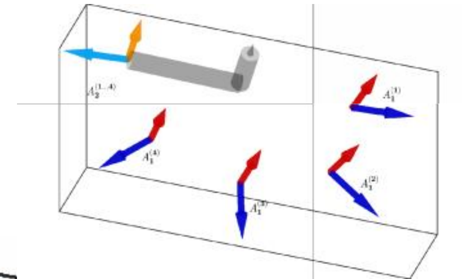

Task parameterized HSMM

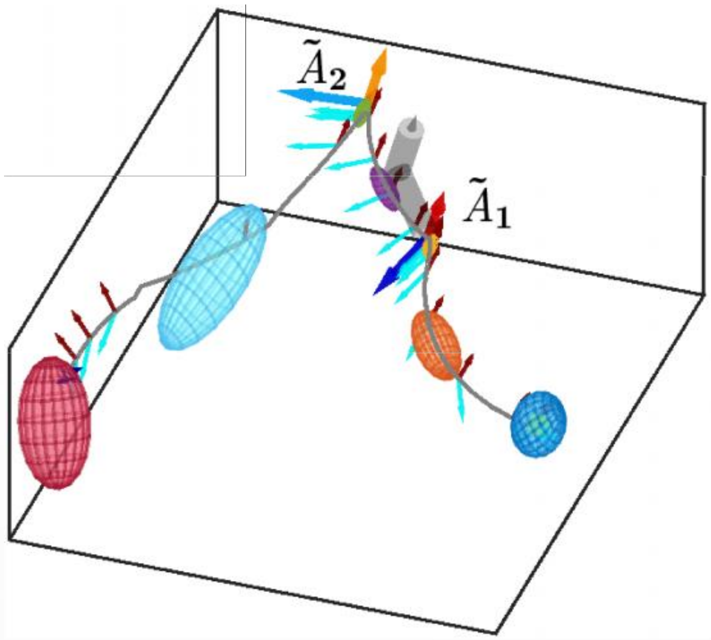



Task parameterized semi-tied HSMM

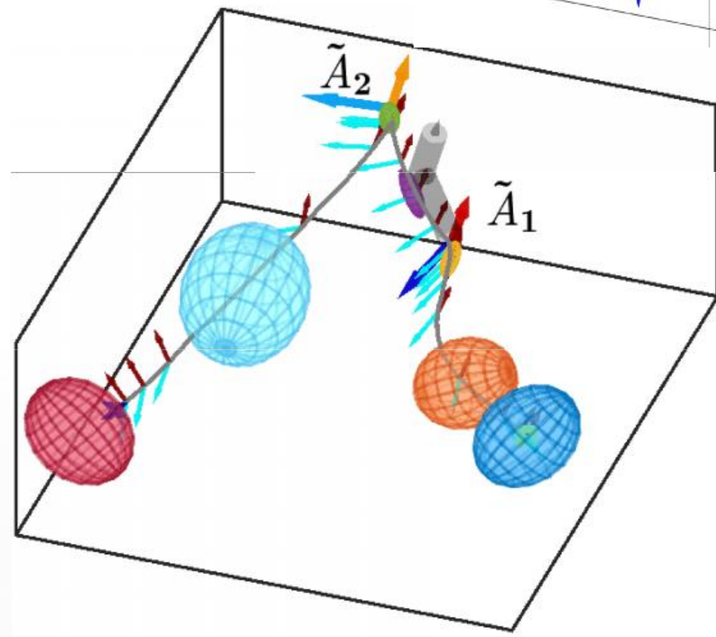
- Task parameterized semi-tied mixture components are better aligned and scaled

# Valve Opening Experiment with Baxter






Task parameterized HSMM




Task parameterized semi-tied HSMM

- Task parameterized semi-tied mixture components are better aligned and scaled

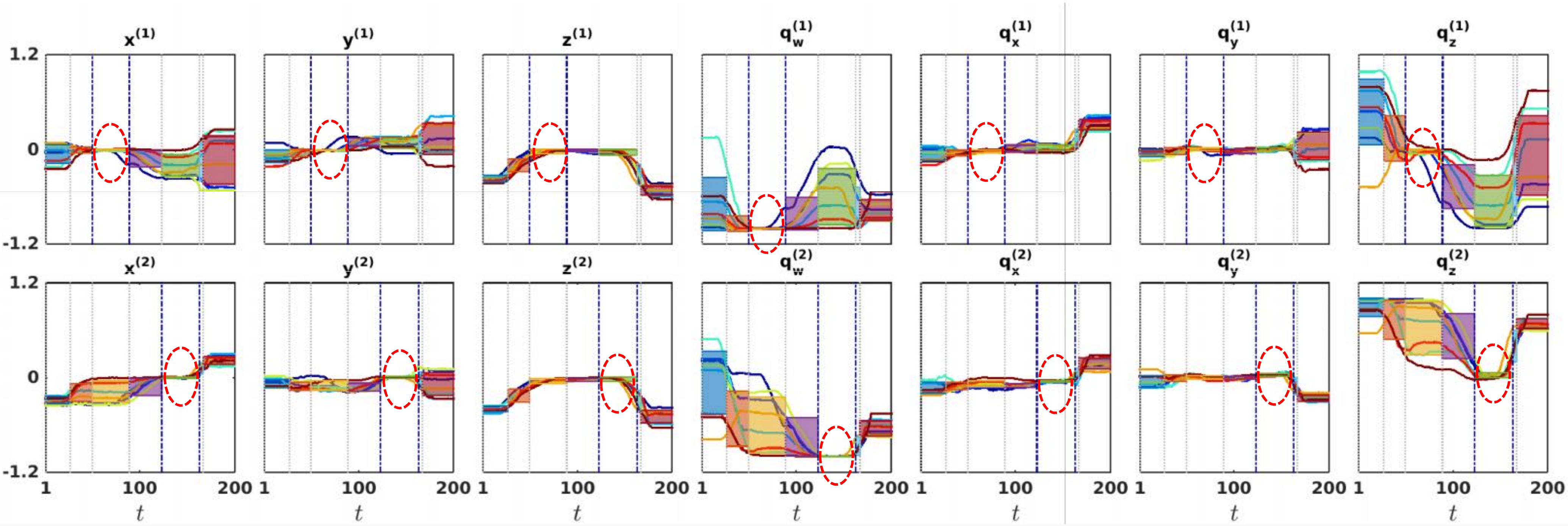
# Valve Opening Experiment with Baxter

| $\alpha$ | Training MSE | Testing MSE | Parameters |
|---|---|---|---|
| 0.0<br>TP-HSMM | **0.0021** | 0.0146 | 1470 |
| 0.5 | 0.0038 | 0.0119 | - |
| 1.0<br>TP-Semi-Tied HSMM | 0.0040 | **0.0119** | **588** |

- Semi-tied model gives better testing accuracy than standard GMM with much less parameters

# Valve Opening Experiment with Baxter

$D = 14, K = 7$




- The model exploits variability in the demonstrations to extract invariant patterns

# Conclusion

- Semi-tied GMMs encode similar coordination patterns with a set of basis vectors /synergistic directions
- Proposed framework combines parsimonious movement representation, task adaptability and optimal control for learning manipulation tasks
- Task-parameterized semi-tied HSMM enables the robot to autonomously deal with different manipulation scenarios in a task